

Professor H. G. Seeley's objection to the importation of the Amphibian plan in explanation of a part of the Plesiosaurian shoulder girdle, of which other parts have been explained by reference to the Chelonian plan of construction, has not, I venture to think, great weight, since, of the early Reptilia, from the time when their remains first began to be studied, it has been a frequent remark that their skeletons comprise structural arrangements which, in existing animals, are now found separately. Moreover, it is thought by some of the ablest comparative anatomists that the Chelonian skeleton shows closer approach to the Amphibian than is to be found elsewhere.

"On Current Curves." By Major R. L. HIPPLISLEY, R.E.  
Communicated by Major MACMAHON, F.R.S. Received  
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(Abstract.)

1. The object of the present paper is to show how to determine expressions for the current in circuits having iron cores, similar to the well-known equations

$$i = \frac{E}{R}(1 - e^{-Rt/L})$$

and

$$i = \frac{E}{\sqrt{(R^2 + p^2 L^2)}} \sin(pt - \theta)$$

for circuits without iron, which will enable the current curves to be pre-determined by calculation and plotted independently of experiment.

In circuits with iron cores the value of  $\frac{dB}{dt}$  occurring in the original differential equations

$$E - \frac{dB}{dt} = Ri \dots \dots \dots (1),$$

$$E \sin pt - \frac{dB}{dt} = Ri \dots \dots \dots (2)$$

continually alters as  $i$  changes. If we could obtain an expression for  $\frac{dB}{dt}$  in terms of  $i$ , the substitution of this expression in (1) and (2) should lead us to the required result. But, though such an expression can be found, its substitution will *generally* lead to differential equations which cannot be solved by known methods.

2. In the case represented by (1), where the applied E.M.F. is constant, we can determine by Lagrange's formula of interpolation

the equation to the (B, H) curve of the particular core under consideration. This will be of the form

$$B = a_0 + a_1 H + a_2 H^2 + \dots + a_n H^n \dots \dots \dots (3),$$

where  $n$  is one less than the number of observed simultaneous values of  $B$  and  $H$  from which the equation is calculated ; whence

$$\frac{dB}{dt} = a_1 L \frac{di}{dt} + 2a_2 L^2 i \frac{di}{dt} + \dots + na_n L^n i^{n-1} \frac{di}{dt} \dots \dots (4),$$

and substituting in the equation

$$E - \frac{dB}{dt} = Ri \dots \dots \dots (5),$$

we get

$$\frac{a_1 L + 2a_2 L^2 i + 3a_3 L^3 i^2 + \dots + na_n L^n i^{n-1}}{E - Ri} di = dt \dots \dots (6),$$

which is easily integrable ; and integrating between the proper limits we get after reduction

$$\begin{aligned} t = \log \frac{E}{E - Ri} & \left( \frac{1}{R} a_1 L + \frac{E}{R^2} 2a_2 L^2 + \frac{E^2}{R^3} 3a_3 L^3 + \dots + \frac{E^{n-1}}{R^n} na_n L^n \right) \\ & - i \left( \frac{1}{R} 2a_2 L^2 + \frac{E}{R^2} 3a_3 L^3 + \frac{E^2}{R^3} 4a_4 L^4 + \dots + \frac{E^{n-2}}{R^{n-1}} na_n L^n \right) \\ & - \frac{1}{2} i^2 \left( \frac{1}{R} 3a_3 L^3 + \frac{E}{R^2} 4a_4 L^4 + \frac{E^2}{R^3} 5a_5 L^5 + \dots + \frac{E^{n-3}}{R^{n-2}} na_n L^n \right) \\ & - \frac{1}{3} i^3 \left( \frac{1}{R} 4a_4 L^4 + \frac{E}{R^2} 5a_5 L^5 + \frac{E^2}{R^3} 6a_6 L^6 + \dots + \frac{E^{n-4}}{R^{n-3}} na_n L^n \right) \\ & - \&c., \text{ to } n+1 \text{ terms} \dots \dots \dots (7). \end{aligned}$$

The corresponding equation when the E.M.F. is removed and the current is dying away is

$$\begin{aligned} t = \frac{f_1 L}{R} \log \frac{E}{Ri} \\ & + \frac{1}{R} \left( 2f_2 L^2 \frac{E}{R} + \frac{1}{2} 3f_3 L^3 \frac{E^2}{R^2} + \frac{1}{3} 4f_4 L^4 \frac{E^3}{R^3} + \dots + \frac{1}{n-1} nf_n L^n \frac{E^{n-1}}{R^{n-1}} \right) \\ & - \frac{1}{R} \left( 2f_2 L^2 i + \frac{1}{2} 3f_3 L^3 i^2 + \frac{1}{3} 4f_4 L^4 i^3 + \dots + \frac{1}{n-1} nf_n L^n i^{n-1} \right) \dots (8), \end{aligned}$$

the  $f_1, f_2, f_3$ , &c., being the coefficients of the powers of  $H$  in the equation to the descending (B, H) curve, which is, of course, different to the ascending curve.

3. This method is not applicable to the case in which the impressed E.M.F. is sinusoidal, on account of difficulties of integration. But both cases can be treated in another way:—Take a series of points on the (B, H) curve of the iron core, such that the chords joining them practically coincide with the curve itself. Let  $B_\kappa$ ,  $H_\kappa$  and  $B_{\kappa+1}$ ,  $H_{\kappa+1}$  be the coordinates of two consecutive points. The equation to the curve between these points is approximately

$$B = m_{\kappa+1}H + \text{constant} \dots\dots\dots (9),$$

where

$$m_{\kappa+1} = \frac{B_{\kappa+1} - B_\kappa}{H_{\kappa+1} - H_\kappa},$$

and therefore between these limits

$$\frac{dB}{dt} = m_{\kappa+1}L \frac{di}{dt} \dots\dots\dots (10).$$

During the time that the current rises from  $i_\kappa$  to  $i_{\kappa+1}$ , and B and H rise from  $B_\kappa$  and  $H_\kappa$  to  $B_{\kappa+1}$  and  $H_{\kappa+1}$ , and  $t$  rises from  $t_\kappa$  to  $t_{\kappa+1}$ , we have

$$E - m_{\kappa+1}L \frac{di}{dt} = Ri \dots\dots\dots (11),$$

and therefore

$$t_{\kappa+1} = t_\kappa + \frac{m_{\kappa+1}L}{R} \log \frac{E - Ri_\kappa}{E - Ri_{\kappa+1}} \dots\dots\dots (12),$$

which is true to a very close approximation for any simultaneous values of  $t$  and  $i$  between the above limits. From this equation, since  $t_0$  and  $i_0$  are both zero, we can determine in succession the times at which the current has the known values  $0, \frac{H_1}{L}, \frac{H_2}{L}, \dots\dots$  &c., using that value of  $m$  which applies to that particular value of H under consideration. In this way the current curve can be plotted.

On making  $E = 0$  in the original differential equation, and observing the proper limits, we get

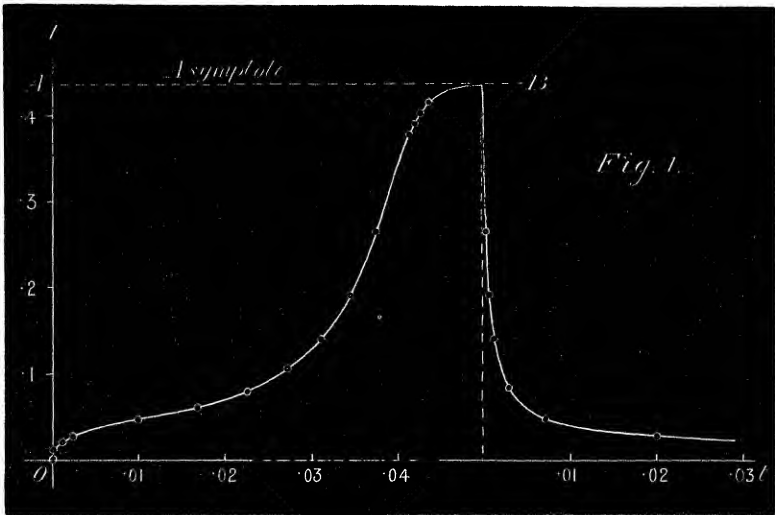
$$t_{n+1} = t_n + \frac{m_{n+1}L}{R} \log \frac{H_n}{H_{n+1}} \dots\dots\dots (13)$$

as the equation to the curve representing the dying away of the current when the E.M.F. is withdrawn;  $m_n, m_{n+1}$  being determined from the descending (B, H) curve.

Fig. 1 and Table I give the results of calculation for a circuit with the following constants:—Resistance, 1 ohm; E.M.F., 0.4315 volt; self-induction (without iron core), 0.0004 henry.

Table I.

Ascending.		Descending.	
Current.	Time.	Current.	Time.
0·0100	0·00023	0·43150	0·000000
0·0200	0·00095	0·26350	0·000235
0·0270	0·00228	0·19050	0·000768
0·0460	0·01025	0·14050	0·001460
0·0600	0·01649	0·08250	0·003036
0·0825	0·02269	0·04600	0·007670
0·1075	0·02696	0·02700	0·020009
0·1400	0·03100	0·01775	0·039880
0·1905	0·03456	0·00925	0·067370
0·2635	0·03768	0·00475	0·101070
0·3545	0·04048	0·00125	0·185770
0·3750	0·04150	0·00000	α
0·3875	0·04205		
0·4000	0·04278		
0·4125	0·04367		
0·4315	α		



4. When the impressed E.M.F. is sinusoidal, we substitute for  $\frac{dB}{dt}$  in the equation

$$E \sin pt - \frac{dB}{dt} = Ri \dots\dots\dots (14),$$

having determined the various values of  $\frac{dB}{dt}$ , as in the foregoing.  
As by the present method the value of  $m$  changes *abruptly* from

$m_\kappa$  to  $m_{\kappa+1}$ , we must employ the *general* solution of (14), which for the interval  $t_\kappa, t_{\kappa+1}$  is

$$i = \frac{E}{\sqrt{(R^2 + m_{\kappa+1}^2 p^2 L^2)}} \sin(pt - \theta_{\kappa+1}) + A_{\kappa+1} e^{-Rt/m_{\kappa+1}L} \dots (15),$$

in order that the current at the commencement of the interval  $t_\kappa, t_{\kappa+1}$  may have the same value which it had at the end of the interval  $t_{\kappa-1} t_\kappa$ . The complementary function

$$A_{\kappa+1} e^{-Rt/m_{\kappa+1}L}$$

enables us to ensure this condition; for, by taking the constant  $A_{\kappa+1}$  of such a value that equation (15) is satisfied when  $i = i_\kappa$  and  $t = t_\kappa$ , there is no abrupt change in the current. The complementary function, in fact, represents the gradual dying away of whatever excess or defect of current there would be in the circuit when  $m$  changes.

Equation (15) is true for all values of  $i$  between  $i_\kappa$  and  $i_{\kappa+1}$ ; and, therefore, enables us to find the time  $t_{\kappa+1}$  at which the current attains the known value  $H_{\kappa+1}/L$ .

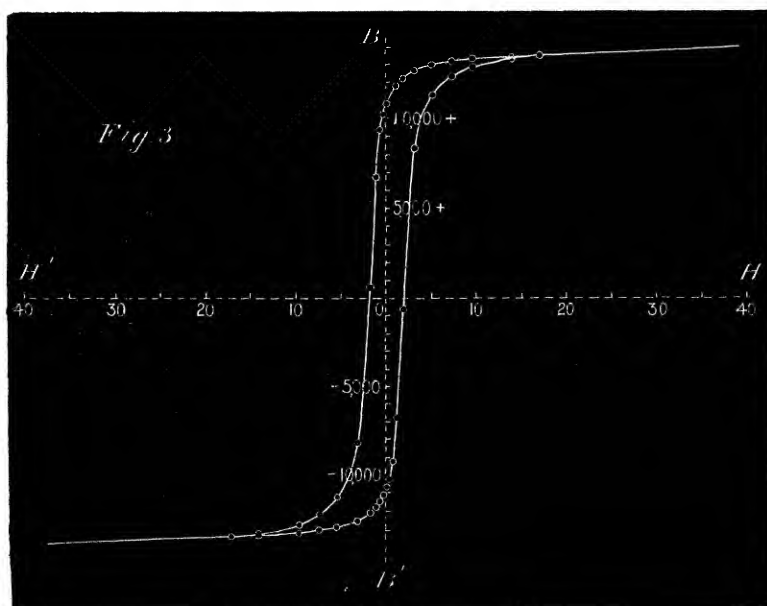
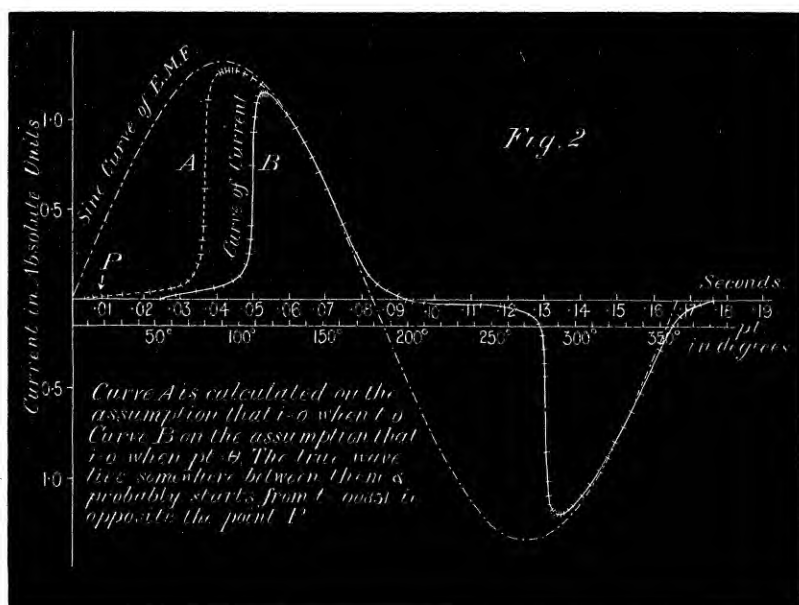
By changing  $\kappa$  into  $\kappa + 1$  we obtain similarly the time  $t_{\kappa+2}$  at which the current has the value  $H_{\kappa+2}/L$ , and so on.

Thus the determination of  $t_{\kappa+1}$  is made to depend upon  $t_\kappa$ , and in order to make a start we must assume that the value of  $i$  is known for some definite value of  $t$ . When the number of alternations per second is not great, it is not of much consequence what assumption, within reason, is made, as, though the calculated curves will vary with the assumption made, they will all eventually merge into the true periodic current curve at some point which will be exhibited when the *first* evanescence of  $Ae^{-Rt/mL}$  takes place.

As this complementary function is a continually decreasing quantity, it becomes negligible when it is allowed time enough. This opportunity is afforded when the straighter portions of the (B, H) curve are reached, and where the points on the curve can be taken further apart.

When, however, the period of alternation is short in comparison with the time-constant of the circuit, the evanescence of  $Ae^{-Rt/mL}$  does not so readily take place; and it will generally take several cycles before the current shakes down into its truly periodic form. The preliminary assumption ought therefore in such cases to be made with care if it is desired to avoid the labour of calculating the first cycles. But, if the periodicity is quick enough,  $Ae^{-Rt/mL}$  can be taken as a constant, at any rate during the shorter intervals.

Fig. 2 gives the plotted curve calculated for a circuit consisting of 500 turns surrounding an anchor ring, having a coefficient of self-induction (without the core) of 0.0004 henry, and a resistance of



1 ohm. The impressed E.M.F. is 12.5 volts. The periodicity is slow, being 6 cycles per second, and the true periodic curve is seen to appear before the end of the first half period. Curve A is calculated with the preliminary supposition that  $i = 0$  when  $t = 0$ , and Curve B on the assumption that  $i = 0$  when  $pt = \theta$ , neither of which is strictly correct.

Fig. 3 gives the (B, H) curve, and Table III the observations upon which it is based, of the iron core under consideration.

5. The method of this paper can be applied to the pre-determination of the curves of the primary and secondary currents in transformers and of the curve of magnetic induction with regard to time.

[For, when there is supposed to be no magnetic leakage in the core, the expressions for the primary and secondary currents are—

$$x = \frac{E\sqrt{(n_1^4 S^2 + p^2 m^2 n_2^4 L^2)}}{D} \sin(pt - \theta) + Ae^{-qt} \dots (16),$$

$$y = -\frac{n_1 n_2 p m L E}{D} \cos(pt - \phi) + Be^{-qt} \dots (17),$$

$$\text{where } \tan \theta = \frac{n_1^4 p m L S^2}{n_1^4 R S^2 + n_2^2 p^2 m^2 L^2 (n_2^2 R + n_1^2 S)},$$

$$\tan \phi = \frac{p m L (n_2^2 R + n_1^2 S)}{n_1^2 R S},$$

$$D = \sqrt{\{n_1^4 R^2 S^2 + p^2 m^2 L^2 (n_2^2 R + n_1^2 S)^2\}}$$

$$q = \frac{n_1^2 R S}{m L (n_2^2 R + n_1^2 S)},$$

R being resistance of primary, S that of secondary,  $n_1$  the number of primary turns,  $n_2$  the number of secondary turns, L the self-induction of the primary.

From (16) and (17) H, the total magnetic force on the core (being  $\frac{n_1 x + n_2 y}{n_1^2} L$ ) reduces to

$$H = \frac{n_1 L S E}{D} \sin(pt - \phi) + Ce^{-qt} \dots (18),$$

$$\text{where } C = \frac{n_1 A + n_2 B}{n_1^2} L.$$

Now equation (18) treated in the same way as (15) gives the simultaneous values of H and  $t$ ; that is, of  $m$  and  $t$ . These latter substituted in (16) and (17) give the simultaneous values of  $x$  and  $t$  and  $y$  and  $t$ .

If there is magnetic leakage it is necessary to discriminate between the  $H$  of the primary core and the  $H$  of the secondary core; these are respectively—

$$H_1 = \frac{E}{n_1} \cdot \frac{\sqrt{\{L^2S^2 + p^2m^2(LN - M^2)^2\}} \sin(pt - \epsilon)}{\sqrt{\{RS - p^2m^2(LN - M^2)\}^2 + p^2m^2(NR + LS)^2}} + C_1e^{-q_1t} + C_2e^{-q_2t} \dots \dots (19),$$

$$\text{and } H_2 = \frac{E}{n_2} \frac{MS \sin(pt - \psi)}{\sqrt{\{RS - p^2m^2(LN - M^2)\}^2 + p^2m^2(NR + LS)^2}} + D_1e^{-q_1t} + D_2e^{-q_2t} \dots \dots (20),$$

$$\text{derived respectively from } H_1 = \frac{1}{n_1} (Lx + My) \text{ and } H_2 = \frac{1}{n_2} (Mx + Ny),$$

$$\text{where } x = \frac{E\sqrt{(S^2 + p^2m^2N^2)} \sin(pt - \eta)}{F} + A_1e^{-q_1t} + A_2e^{-q_2t} \dots \dots (21),$$

$$\text{and } y = -\frac{EpmM \cos(pt - \psi)}{F} + B_1e^{-q_1t} + B_2e^{-q_2t} \dots \dots \dots (22).$$

In these equations  $N$  is the self-induction of the secondary,  $M$  is the mutual induction,  $F$  is the radicle in the denominator of (19) and (20), while  $\tan \eta = pm \frac{LS^2 - p^2m^2N(LN - M^2)}{R(S^2 + p^2m^2N^2) + p^2m^2MS}$ ,

$$\tan \psi = pm \frac{NR + LS}{RS - p^2m^2(LN - M^2)},$$

$$\tan \epsilon = pm \frac{L^2S^2 + RSM^2 + p^2m^2(LN - M^2)^2}{R\{LS^2 + p^2m^2N(LN - M^2)\}},$$

$$q_1 = \frac{NR + LS - \sqrt{\{(NR - LS)^2 + 4RSM^2\}}}{2m(LN - M^2)},$$

$$q_2 = \frac{NR + LS + \sqrt{\{(NR - LS)^2 + 4RSM^2\}}}{2m(LN - M^2)}.$$

September, 1892.]



Table IIa.—Calculation of First Half Periods, assuming  $i = 0$  when  $t = 0$ .  
(The \* shows the point at which this coalesces with Table IIb.)

Ascending.		Descending.		Portion showing evanescence of $Ae^{-Rt/mL}$ .				
Current.	Time.	Current.	Time.	$Ae^{-Rt/mL}$ .	$E \sin (pt - \theta) / \sqrt{(R^2 + m^2 p^2 L^2)}$ .	Current.	Time.	
0·00000	0·000000	1·24970	0·042450	0·1175	1·2305	1·1180	0·03782	
0·00475	0·002799	0·35475	*0·076483	0·0060	1·2450	1·2390	0·04013	
0·00925	0·004837	0·26350	0·078920	0·0040	1·2465	1·2435	0·04060	
0·01775	0·008255	0·14050	0·082320	0·0020	1·2480	1·2460	0·04106	
0·02700	0·013485	0·08250	0·084340	0·0010	1·2490	1·2480	0·04152	
0·04600	0·021850	0·04600	0·087050	0·0003	1·2500	1·2497	0·04245	
0·06250	0·027197	0·01775	0·089850	0·0001	1·2495	1·2494	0·04291	
0·08250	0·030850	0·00000	0·091840	0·0000	1·2490	1·2490	0·04337	
0·10750	0·032550			becomes	1·2480	1·2480	0·04384	
0·14050	0·033747			negligible	1·2465	1·2465	0·04430	
0·19075	0·035264				1·2450	1·2450	0·04476	
0·26350	0·035880				1·2305	1·2305	0·04708	
0·35475	0·036060				1·2070	1·2070	0·04939	
0·43150	0·036310				1·1770	1·1770	0·05171	
1·24970	0·042450				1·1330	1·1330	0·05402	
			True current wave starts from this point.					
						1·0820	1·0820	0·05694
						1·0235	1·0235	0·05865
						0·9570	0·9570	0·06097
						0·8835	0·8835	0·06328
						0·8030	0·8030	0·06560
						0·7165	0·7165	0·06791
						0·6245	0·6245	0·07023
						0·5280	0·5280	0·07254
						0·4275	0·4275	0·07486
					*0·3547	0·3547	0·07648	

Table IIb.—Calculation of First Half Periods, assuming  $i = 0$  when  $pt = \theta$ .  
(The \* shows the point at which this coalesces with Table IIa.)

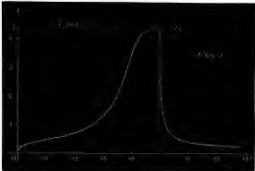
Ascending.		Descending.		Portion showing erascescence of $Ae^{-Rt/_{mL}}$ .			
Current.	Time.	Current.	Time.	$Ae^{-Rt/_{mL}}$ .	$\frac{E \sin (pt - \theta)}{\sqrt{(R^2 + m^2 p^2 L^2)}}$	Current.	Time.
0.00000	0.025470	1.13700	0.053100	0.45600	1.1950	0.7390	0.05032
0.00475	0.025640	0.35475	*0.070483	0.25200	1.1885	0.9365	0.05078
0.00925	0.025975	0.26350	0.078920	0.13950	1.1815	0.9420	0.05124
0.01775	0.026670	0.14050	0.082320	0.07700	1.1740	1.0970	0.05171
0.02700	0.029020	0.08250	0.084340	0.04250	1.1665	1.1240	0.05217
0.04600	0.034470	0.04000	0.087050	0.02350	1.1585	1.1350	0.05263
0.06250	0.038800	0.01775	0.089850	0.01300	1.1500	1.1370	0.05310
0.08250	0.042900	0.00000	0.091840	0.00700	1.1415	1.1345	0.05356
0.10750	0.045810			0.00400	1.1330	1.1290	0.05402
0.14050	0.046960			0.00200	1.1235	1.1215	0.05449
0.19075	0.048640			0.00100	1.1135	1.1125	0.05495
0.26350	0.049256			0.00050	1.1035	1.1030	0.05541
0.35475	0.049670			0.00030	1.0930	1.0927	0.05587
0.43150	0.049910		True current wave starts from this point.				
1.13700	0.053100			0.00000 becomes negligible	1.0820	1.0820	0.05634
					1.0235	1.0235	0.05865
					0.9570	0.9570	0.06097
					0.8835	0.8835	0.06328
					0.8030	0.8030	0.06560
					0.7165	0.7165	0.06791
					0.6245	0.6245	0.07023
					0.5280	0.5280	0.07254
					0.4274	0.4274	0.07486
					0.4068	0.4068	0.07532
					0.3861	0.3861	0.07578
					0.3653	0.3653	0.07624
					*0.3547	0.3547	0.07648



Table III.—Observations of (B, H) Curve.\*

First half cycle.		Second half cycle.		Third half cycle.		Loop.	
H.	B.	H.	B.	H.	B.	H.	B.
0·00	0	0·00	+10,980	0·00	-10,800	0·00	+10,844
+ 0·40	100	- 0·05	+10,970	+ 0·05	-10,795	+ 0·05	+10,845
+ 0·80	+ 400	- 0·19	+10,790	+ 0·19	-10,620	+ 0·19	+10,850
+ 1·08	+ 941	- 0·37	+10,460	+ 0·37	-10,268	+ 0·37	+10,860
+ 1·84	+ 4,088	- 0·71	+ 9,530	+ 0·71	- 9,270	+ 0·71	+10,890
+ 3·30	+ 8,684	- 1·08	+ 7,360	+ 1·08	- 6,780	+ 1·08	+10,930
+ 5·62	+11,380	- 1·84	+ 20	+ 1·84	- 642	+ 1·84	+11,040
+ 7·63	+12,320	- 3·30	- 7,840	+ 3·30	+ 8,190	+ 3·30	+11,480
+10·54	+12,950	- 5·62	-11,100	+ 5·62	+11,270	+ 5·62	+12,210
+14·19	+13,280	- 7·63	-12,080	+ 7·63	+12,170	+ 7·63	+12,650
+17·26	+13,450	-10·54	-12,690	+10·54	+12,770	+10·54	+13,040
+14·19	+13,370	-14·19	-13,190	+14·19	+13,110	+14·19	+13,340
+10·54	+13,250	-17·26	-13,110	+17·26	+13,280	+17·26	+13,440
+ 7·63	+13,100	-14·19	-13,000	+14·19	+13,220		
+ 5·62	+12,970	-10·54	-12,800	+10·54	+13,100		
+ 3·30	+12,680	- 7·63	-12,900	+ 7·63	+12,940		
+ 1·84	+12,240	- 5·62	-12,800	+ 5·62	+12,880		
+ 1·08	+11,850	- 3·30	-12,520	+ 3·30	+12,570		
+ 0·71	+11,590	- 1·84	-12,080	+ 1·84	+12,150		
+ 0·37	+11,277	- 1·08	-11,700	+ 1·08	+11,790		
+ 0·19	+11,100	- 0·71	-11,430	+ 0·71	+11,510		
+ 0·05	+11,000	- 0·37	-11,130	+ 0·37	+11,200		
0·00	+10,980	- 0·19	-10,970	+ 0·19	+11,080		
		- 0·05	-10,820	+ 0·05	+10,880		
		0·00	-10,800	0·00	+10,840		

\* Taken from Professor Ewing's "Magnetism in Iron, &c.," Phil. Trans., 1885, except the first three observations, which have been made to show the iron starting without initial magnetism. H and B are in absolute units per sq. cm.



Frequency in thousands cycles

Fig. 2

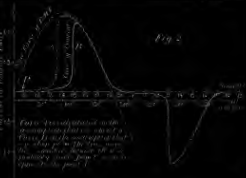


Figure 2 is a graph of the frequency in thousands cycles versus time in microseconds. The curve labeled A is the curve of the frequency in thousands cycles versus time in microseconds. The curve labeled B is the curve of the frequency in thousands cycles versus time in microseconds. The area between the two curves is labeled 'Area A' and 'Area B'. The area under Curve A is labeled 'Area A' and the area under Curve B is labeled 'Area B'.

Figure 2

